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Multilayer Parking with Screening on a Random Tree

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Abstract In this paper we present a multilayer particle-deposition model on a random tree. We derive the time-dependent densities of the first and second layer analytically and show that for all trees the limiting density of the first layer exceeds the density in the second layer. We also provide a procedure to calculate higher-layer densities and prove that random trees have a higher limiting density in the first layer than regular trees. Finally, we compare densities between the first and second layer and between regular and random trees.

Keywords Car parking problem · Random sequential adsorption · Sequential frequency assignment process · Particle systems

1 Introduction

Parking models with screening were first studied in the field of ballistic particle deposition, see for example [1]. In those models particles are moving towards a substrate or a fiber until they encounter a previously deposited particle or the substrate itself. A particle always tries to park on a layer as low as possible but due to "screening" the particle cannot pass formerly deposited particles. In our model the screening rule makes every particle park in the highest layer possible where it is supported by a particle in the layer below (see Fig. 1). Models of this type provide an interesting class of non-equilibrium systems; physically they describe the addition of particles or atoms to surfaces in regimes where diffusion on the substrate does not play a role on the experimental time-scale. This "Tetris" model is very different from the so-called Sequential Frequency Assignment Process (SFAP) in which particles (assignments) can skip particles on their way down [2]. In that model particles are deposited

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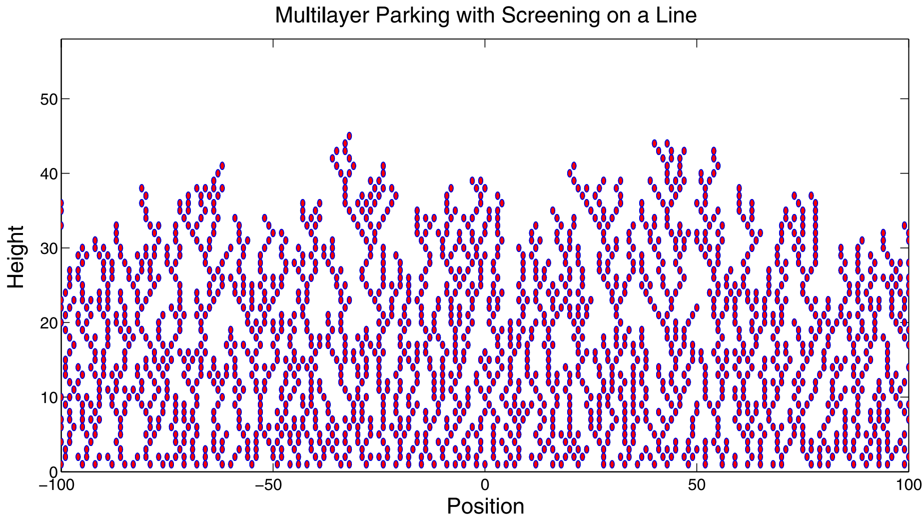

Fig. 1 Example of a realization of the particle deposition process with screening on a one-dimensional lattice

in the lowest layer (frequency) possible. It has been found for the SFAP that there is an increasing limiting density of particles in higher layers due to boundary and other effects [2, 3]. In this paper we will show analytically that in the model with screening the opposite is true. The density in the first layer turns out to be higher than in the second layer. We conjecture that the same applies to the other layers.

Also, we generalize the model on the one-dimensional lattice to a model on regular and random trees. Recently, several random particle-deposition tree models have been studied in [4–6]. However, to our knowledge this is the first time a multilayer random tree model is treated.

2 Layer Densities

2.1 The Dynamics

The precise definition of the model is as follows. We consider a random tree with vertices $i \in V$ and degree at the site i given by D_i. We choose D_i to be independent random variables with the same distribution $\mathbb{Q}$ given by

$$\mathbb{Q}(D_i = k) = a_k \tag{2.1}$$

on the integers starting from 2. The latter requirement ensures that we have no open ends with probability one.

We denote the generating function of the distribution by

$$G(s) = \sum_{k=2}^{\infty} a_k s^k \tag{2.2}$$

We will denote the expected value with respect to this probability distribution by the same symbol $\mathbb{Q}$. We fix a realization of the random tree, and denote by V its vertex set.

In our deposition model particles will be dropped at sites i according to Poisson processes with rate 1, independently over the sites. When a particle arrives at site i it will be deposited at height $\max\{h_j, \mathrm{dist}(j,i) \le 1\} + 1$ where h_j is the maximum height at which a particle is present at site j. So, the resulting configurations at any given time are such that there can be no neighboring particles in the same layer, and any deposited particle is supported by a particle directly below it or at a horizontal distance one below it (see Fig. 1).

In the first part of the paper we will be interested in the behavior of particle configurations arising from particle deposition in the first two layers, that is we consider the marginal of an infinite-layer particle model on the first two layers. This will be generalized to higher layers later. To describe the behavior on the first two layers we consider (suitably coded) occupation numbers $m = (m(i))_{i\in V} \in \Omega = \{0,1,2,3\}^V$. Here the *spin* $m(i)$ denotes the joint occupation numbers at vertex i at height 1 and 2. It is useful for short notation to interpret the occupation numbers at various heights as binary digits and write ordinary natural numbers. That is we write

$$m(i) = \begin{cases} 0 & \text{if vertex } i \text{ is vacant in the first and second line} \\ 1 & \text{if vertex } i \text{ is occupied in the first but not in the second line} \\ 2 & \text{if vertex } i \text{ is occupied in the second but not in the first line} \\ 3 & \text{if vertex } i \text{ is occupied in the first and in the second line} \end{cases} \tag{2.3}$$

so that $m(i) \in \{0,1,2,3\}$.

Now we can describe the dynamics of the joint process of particle occupations in the first two layers and the total number of particles which have arrived by the following generator. Note that it is really necessary to consider also particle arrivals beyond the first two layers because of the screening effects higher-layer particles might have on lower layers. It is furthermore necessary to distinguish two sorts of particle arrivals: those which change the lower layers and those which leave the lower layers unchanged. Let F be a joint function of particle occupations in the first two layers and particle numbers. Then the generator of our process reads

$$\begin{aligned} \mathcal{L}F(m,N) = \sum_{k\in V}\Bigg(&\sum_{s=1,2,3,4} F(m^{s,k}, N^k) r_k(s; \mathfrak{M}_k) \\ &+ \Bigg(1 - \sum_{s=1,2,3,4} r_k(s;\mathfrak{M}_k)\Bigg) F(m, N^k) - F(m,N)\Bigg) \end{aligned} \tag{2.4}$$

with

$$N^k(i) = \begin{cases} N(k) + 1 & \text{if } k = i \\ N(k) & \text{if } k \ne i \end{cases} \tag{2.5}$$

and with

$$m^{s,k}(i) = \begin{cases} s & \text{if } k = i \\ m(i) & \text{if } k \ne i \end{cases} \tag{2.6}$$

where $\mathfrak{M}_k := (N(l), m(l))_{l\in\{k\}\cup C(k)}$ where $C(k) := \{i : \mathrm{dist}(i,k) = 1\}$ is the neighborhood of vertex k.

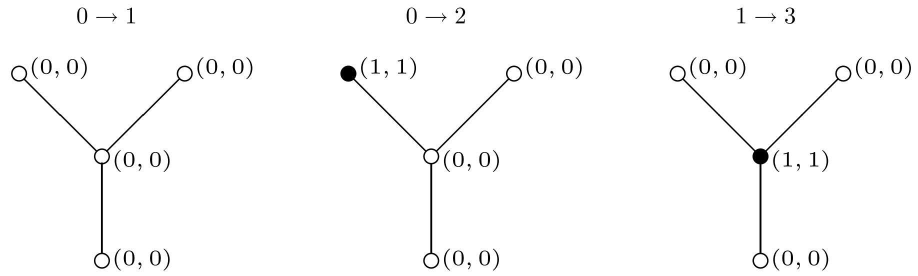

Fig. 2 Neighborhood configurations of a vertex that allow the m_t transitions $0 \to 1$, $1 \to 2$ and $1 \to 3$ respectively. In this example the central vertex has three neighbors. The states are denoted with the notation (N_t, m_t), e.g. $(1,1)$ means that one particle arrived and that it was deposited on the first layer. In order to have a transition from 0 to 1 every vertex in the neighborhood has to be totally empty ($N_t = 0$). To get a transition from 0 to 2 the vertex itself must be empty, but at least one of the neighbors has to have one particle in total that lies on the first layer. Finally, to have a transition 1 to 3 the vertex has to have exactly one particle located on the first layer, while the neighbors should be empty

Looking at test functions F which do not depend on the m-variable we see that this generator will give rise to Poisson-processes $N_t(k)$, independently over the sites k. To understand the form of the generator describing the particle fillings which are encoded by the m-variable we note that the first term in the generator describes the events when the addition of a new particle also changes the configuration in one of the first two layers. The second term in the generator describes the events when the first two layers are already full or screened, and a further adding of a particle does not change its filling.

The rates are either equal to zero or one. They are 1 precisely in the following cases (see also Fig. 2) listed below.

1. $0 \mapsto 1$ Adding a particle in the first line at vertex i. We have

$$r(1; \{N(i) = 0\} \cap \{\forall_{j \in C(i)} : N(j) = 0\}) = 1 \tag{2.7}$$

   Indeed, this occurs when the site and all its neighbors are empty in all layers.

2. $0 \mapsto 2$ Adding a particle in the second line at i while the first line was empty at the site

$$r(2; \{N(i) = 0, \exists J \subset C(i) : \forall j \in J : m(j) = 1, N(j) = 1, \forall k \in C(i) \backslash J : N(k) = 0\}) = 1 \tag{2.8}$$

   It is only possible to reach the state $m(i) = 2$ when there is a non-empty set of neighboring sites which have a particle at layer 1 and where no more than one particle has been dropped, while in all other neighboring sites no particles have been dropped.

3. $1 \mapsto 3$ Adding a particle in the second line while the first line was full at the site

$$r(3; \{m(i) = 1, N(i) = 1\} \cap \{\forall_{j \in C(i)} : N(j) = 0\}) = 1 \tag{2.9}$$

   To get into state $m(i) = 3$ there must be one particle in vertex i and all neighboring sites should be empty to avoid screening.

   All other transitions are impossible.

This generator defines a time-homogeneous Markov jump process on the infinite graph by standard theory [7] such that (2.4) $\frac{d}{dt}|_{t=0} \mathbb{E}^{m,N} F(m_t, N_t) = \mathcal{L}F(m, N)$.

Here $\mathbb{E}^{m,N}$ denotes the expected value with respect to the process, started in the initial configuration $(m, N) = (m(i), N(i))_{i \in V}$ at $t = 0$.

We underline that we consider the marginal of the first two layers of a model where particles may pile up to arbitrarily high layers. The present model differs from the model discussed in [3] where particles that cannot be deposited in the first or second layer are rejected.

2.2 Regular Trees

We first consider the densities, taken at an arbitrary vertex called 0,

$$\begin{aligned}\rho_t^d(1) &= P_t(m(0)=1) + P_t(m(0)=3)\\ \rho_t^d(2) &= P_t(m(0)=2) + P_t(m(0)=3)\end{aligned} \tag{2.10}$$

on the first and second layer on a regular tree with degree $d \geq 2$. Having understood their behavior on a regular tree we can derive the densities on random trees easily in Sect. 2.3.

Theorem 1 *Consider the regular tree T_d with degree $d \geq 2$. Particles arrive at the vertices of T_d according to a Poisson process and obey the screening rules of deposition. Then the time-dependent densities are, on the first layer*

$$\rho_t^d(1) = \frac{1 - e^{-(d+1)t}}{d+1} \tag{2.11}$$

and on the second layer

$$\begin{aligned}\rho_t^d(2) = {}& \left(\frac{d}{d-1}\right)^d \sum_{k=0}^{d} \binom{d}{k} \frac{d^{-k}(-1)^k}{(d-1)k+d+1} - \frac{d}{(d+1)^2} + \frac{d}{(d+1)^2} e^{-(d+1)t}\\ &- \left(\frac{d}{d-1}\right)^d \sum_{k=0}^{d} \binom{d}{k} \frac{d^{-k}(-1)^k}{(d-1)k+d+1} e^{-[(d-1)k+d+1]t} - \frac{1}{d+1} t e^{-(d+1)t}\end{aligned} \tag{2.12}$$

Proof In this section and the next probabilities regarding the filling in the first two layers, but also regarding the total number of particle arrivals at a site, are introduced. It will be necessary to keep also the latter information; looking at the first type of quantities only does not provide a closed system of equations. Throughout the paper we use the notation

$$D_t^d(s) = P_t(m(0) = s)$$

for all s. We fix at a certain vertex 0. The surrounding vertices are numbered $1, 2, \ldots, d$. First we calculate the time derivative of $D_t^d(1)$ and integrate back. Taking into account the first and third process depicted in Fig. 2 we see that

$$\begin{aligned}\dot{D}_t^d(1) &= P_t(N_t(0)=0, \forall_{k\leq d} N_t(k)=0) - P_t(N_t(0)=1,\ \forall_{k\leq d} N_t(k)=0)\\ &= e^{-(d+1)t} - te^{-t}e^{-dt}\\ &= -(t-1)e^{-(d+1)t}\end{aligned} \tag{2.13}$$

So, now with $D_0^d(1) = 0$ we find

$$D_t^d(1) = \frac{d}{(d+1)^2} - \frac{d}{(d+1)^2} e^{-(d+1)t} + \frac{1}{d+1} t e^{-(d+1)t} \tag{2.14}$$

We apply the same technique to find $D_t^d(2)$

$$
\begin{aligned}
\dot{D}_t^d(2) &= \sum_{k=1}^{d} \binom{d}{k} P_t(N_t(0)=0, \forall_{i\le k} N_t(i)=1, m_t(i)=1, \forall_{k<j\le d} N_t(j)=0) \\
&= \sum_{k=1}^{d} \binom{d}{k} P_t(\forall_{i\le k} N_t(i)=1, m_t(i)=1, \forall_{k<j\le d} N_t(j)=0 | N_t(0)=0) e^{-t} \\
&= e^{-t} \sum_{k=1}^{d} \binom{d}{k} [P_t(N_t(1)=1, m_t(1)=1 | N_t(0)=0)]^k [P_t(N_t(0)=0)]^{d-k} \qquad (2.15)
\end{aligned}
$$

Now we need to calculate the quantity $S_t^d := P_t^d(N_t(1)=1, m_t(1)=1 | N_t(0)=0)$. This is done by constructing another differential equation.

$$
\begin{aligned}
\dot{S}_t^d &= P_t(N_t(1)=0, \forall_{i:\mathrm{dist}(i,1)=1} N_t(i)=0 | N_t(0)=0) - S_t^d \\
&= e^{-dt} - S_t^d \qquad (2.16)
\end{aligned}
$$

whose solution is

$$
S_t^d = e^{-t} \frac{1-e^{-(d-1)t}}{d-1} \qquad (2.17)
$$

So, we have

$$
\begin{aligned}
\dot{D}_t^d(2) &= e^{-t} \sum_{k=1}^{d} \binom{d}{k} \left[\frac{1}{d-1} e^{-t}(1-e^{-(d-1)t}) \right]^k [e^{-t}]^{d-k} \\
&= e^{-(d+1)t} \sum_{k=1}^{d} \binom{d}{k} \left[\frac{1-e^{-(d-1)t}}{d-1} \right]^k \\
&= e^{-(d+1)t} \left[\left(1 + \frac{1-e^{-(d-1)t}}{d-1} \right)^d - 1 \right] \\
&= -e^{-(d+1)t} + e^{-(d+1)t} \left(\frac{d}{d-1} - \frac{1}{d-1} e^{-(d-1)t} \right)^d \\
&= -e^{-(d+1)t} + e^{-(d+1)t} \sum_{k=0}^{d} \binom{d}{k} \left(\frac{d}{d-1} \right)^{d-k} \left(\frac{-1}{d-1} \right)^k e^{-(d-1)kt} \\
&= -e^{-(d+1)t} + \left(\frac{d}{d-1} \right)^d \sum_{k=0}^{d} \binom{d}{k} d^{-k} (-1)^k e^{-[(d-1)k+d+1]t} \qquad (2.18)
\end{aligned}
$$

which gives (with $D_0^d(2)=0$)

$$
\begin{aligned}
D_t^d(2) &= \left(\frac{d}{d-1} \right)^d \sum_{k=0}^{d} \binom{d}{k} \frac{d^{-k}(-1)^k}{(d-1)k+d+1} - \frac{1}{d+1} \\
&\quad + \frac{1}{d+1} e^{-(d+1)t} - \left(\frac{d}{d-1} \right)^d \sum_{k=0}^{d} \binom{d}{k} \frac{d^{-k}(-1)^k}{(d-1)k+d+1} e^{-[(d-1)k+d+1]t} \qquad (2.19)
\end{aligned}
$$

Finally, for $D_t^d(3)$ we find

$$\dot{D}_t^d(3) = P_t(N_t(0) = 1, \forall_{k \le d} N_t(k) = 0)$$
$$= te^{-(d+1)t} \tag{2.20}$$

so, that (with $D_0^d(3) = 0$)

$$D_t^d(3) = \frac{1}{(d+1)^2} - \frac{1}{(d+1)^2} e^{-(d+1)t} - \frac{1}{d+1} te^{-(d+1)t} \tag{2.21}$$

The densities of the first and second layer follow immediately by adding $D_t^d(1)$ and $D_t^d(3)$ for the first layer, and $D_t^d(2)$ and $D_t^d(3)$ for the second layer. □

Remark Note that for the derivation of the formula for the first layer we did not have to use the absence of loops in a tree. Therefore, the first-layer density on a graph is the same as on a tree, no matter whether they are regular or random.

2.3 Random Trees

Let us now consider the case of particle deposition on a random tree where the number of neighbors of every vertex is a random number according to some $G(s) = \sum_{n=2}^{\infty} a_n s^n$. We now have the following

Theorem 2 *Consider a multilayer random tree T_D with generating function $G_T(s) = \sum_{n=2}^{\infty} a_n s^n$. Particles arrive at the vertices of T_D according to a Poisson process and obey the screening rules of deposition. Then the tree-averaged time dependent densities are, on the first layer*

$$\mathbb{Q}\rho_t(1) = \sum_{k=2}^{\infty} a_k \frac{(1 - e^{-(k+1)t})}{k+1} \tag{2.22}$$

and on the second layer

$$\mathbb{Q}\rho_t(2) = \sum_{k=2}^{\infty} \frac{a_k}{(k+1)^2} - \sum_{k=2}^{\infty} \left(\frac{a_k}{(k+1)^2} e^{-(k+1)t} + \frac{a_k}{k+1} te^{-(k+1)t} \right) \tag{2.23}$$

$$+ \sum_{d_0=2}^{\infty} a_{d_0} \sum_{k=1}^{d_0} \binom{d_0}{k} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \left(\sum_{d=2}^{\infty} \frac{a_d}{d-1} \right)^{k-i} \int_0^t Z_u^i e^{-(d_0+1)u} du \tag{2.24}$$

where $Z_t := \sum_{d=2}^{\infty} a_d e^{-(d-1)t}/(d-1)$.

Proof First, we calculate $\mathbb{Q}D_t(1)$ and $\mathbb{Q}D_t(3)$. Notice that the derivatives of these functions in a certain vertex 0 are not affected by the tree ensemble beyond the nearest neighbors. Therefore, we can immediately start averaging $D_t(1)$ and $D_t(3)$ over $\mathbb{Q}$ rather than dealing with its derivatives first. In the previous section we already found

$$D_t^d(1) = \frac{d}{(d+1)^2} - \frac{d}{(d+1)^2} e^{-(d+1)t} + \frac{1}{d+1} te^{-(d+1)t} \tag{2.25}$$

where d now denotes the (random) number of nearest neighbors of the site under consideration. Averaging over $\mathbb{Q}$ results then in

$$\mathbb{Q}D_t(1) = \sum_{k=2}^{\infty} \frac{a_k k}{(k+1)^2} - \sum_{k=2}^{\infty} \left(\frac{a_k k}{(k+1)^2} e^{-(k+1)t} - \frac{a_k}{k+1} t e^{-(k+1)t} \right) \tag{2.26}$$

Similarly, we find

$$\mathbb{Q}D_t(3) = \sum_{k=2}^{\infty} \frac{a_k}{(k+1)^2} - \sum_{k=2}^{\infty} \left(\frac{a_k}{(k+1)^2} e^{-(k+1)t} + \frac{a_k}{k+1} t e^{-(k+1)t} \right) \tag{2.27}$$

Adding these two results gives the density on the first layer. In the previous section we already found

$$\dot{D}_t(2) = e^{-t} \sum_{k=1}^{d_0} \binom{d_0}{k} [P_t^{d_k}(N_t(k) = 1, m_t(1) = 1 | N_t(0) = 0)]^k [P_t(N_t(0) = 0)]^{d_0 - k} \tag{2.28}$$

where the number of neighbors of vertex i is denoted by d_i. Note that in this section the d_i's may be different since we are treating a random tree. So, we get

$$\dot{D}_t(2) = e^{-t} \sum_{k=1}^{d_0} \binom{d_0}{k} [S_t^{d_k}]^k [P_t(N_t(0) = 0)]^{d_0 - k} \tag{2.29}$$

$$\Rightarrow \quad \mathbb{Q}\dot{D}_t(2) = e^{-t} \sum_{d_0=2}^{\infty} a_{d_0} \sum_{k=1}^{d_0} \binom{d_0}{k} \left[\sum_{d_k=2}^{\infty} a_{d_k} S_t^{d_k} \right]^k [P_t(N_t(0) = 0)]^{d_0 - k} \tag{2.30}$$

In (2.17) we already found that $S_t^{d_k} = \frac{1}{d_k - 1} e^{-t}(1 - e^{-(d_k - 1)t})$. So, we have

$$\begin{aligned}
\mathbb{Q}\dot{D}_t(2) &= \sum_{d_0=2}^{\infty} a_{d_0} e^{-(d_0+1)t} \sum_{k=1}^{d_0} \binom{d_0}{k} \left[\sum_{d=2}^{\infty} a_d \frac{1}{d-1} - \sum_{d=2}^{\infty} a_d \frac{e^{-(d-1)t}}{d-1} \right]^k \\
&= \sum_{d_0=2}^{\infty} a_{d_0} \sum_{k=1}^{d_0} \binom{d_0}{k} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \left(\sum_{d=2}^{\infty} a_d \frac{1}{d-1} \right)^{k-i} \\
&\quad \times \left(\sum_{d=2}^{\infty} a_d \frac{e^{-(d-1)t}}{d-1} \right)^i e^{-(d_0+1)t} \\
&= \sum_{d_0=2}^{\infty} a_{d_0} \sum_{k=1}^{d_0} \binom{d_0}{k} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \left(\sum_{d=2}^{\infty} \frac{a_d}{d-1} \right)^{k-i} Z_t^i e^{-(d_0+1)t} \qquad (2.31)
\end{aligned}$$

with $Z_t := \sum_{d=2}^{\infty} a_d e^{-(d-1)t}/(d-1)$. So, by integration we find

$$\mathbb{Q}D_t(2) = \sum_{d_0=2}^{\infty} a_{d_0} \sum_{k=1}^{d_0} \binom{d_0}{k} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \left(\sum_{d=2}^{\infty} a_d \frac{1}{d-1} \right)^{k-i} \int_0^t Z_u^i e^{-(d_0+1)u} du \tag{2.32}$$

□

Example We would like to give an example where a closed-form solution is available which is free of integrals and gives us the time-dependent behavior of densities in the first and second line as sums whose main terms are exponentials in the time. Let us consider the special case where there are only two possible numbers of neighbors a and b on the random tree, i.e. we take $G(s) = p_a s^a + p_b s^b$. We find

$$\mathbb{Q}\rho_t(1) = \frac{p_a}{a+1}\left(1 - e^{-(a+1)t}\right) + \frac{p_b}{b+1}\left(1 - e^{-(b+1)t}\right) \tag{2.33}$$

For the second layer we need to calculate the quantity $C_t(n,x) := \int_0^t Z_u^n e^{-(x+1)u} du$. We have

$$\begin{aligned} Z_t^n &= \left[\frac{p_a}{a-1} e^{-(a-1)t} + \frac{p_b}{b-1} e^{-(b-1)t}\right]^n \\ &= \sum_{j=0}^{n} \binom{n}{j} \left(\frac{p_a}{a-1}\right)^{n-j} \left(\frac{p_b}{b-1}\right)^j e^{-[(a-1)(n-j)+(b-1)j]t} \end{aligned} \tag{2.34}$$

and so

$$C_t(n,x) = \begin{cases} \frac{1}{x+1}(1 - e^{-(x+1)t}) & \text{if } n = 0 \\ \sum_{j=0}^{n} \binom{n}{j} \frac{(\frac{p_a}{a-1})^{n-j} (\frac{p_b}{b-1})^j (1 - e^{-[(a-1)(n-j)+(b-1)j+x+1]t})}{(a-1)(n-j)+(b-1)j+x+1} & \text{if } n > 0 \end{cases} \tag{2.35}$$

So, for the second layer's density we find the closed form

$$\begin{aligned} \mathbb{Q}\rho_t(2) = p_a &\left(\frac{1}{(a+1)^2} - \frac{e^{-(a+1)t}}{(a+1)^2} - \frac{te^{-(a+1)t}}{a+1}\right) + p_b\left(\frac{1}{(b+1)^2} - \frac{e^{-(b+1)t}}{(b+1)^2} - \frac{te^{-(b+1)t}}{b+1}\right) \\ &+ p_a \sum_{k=1}^{a} \binom{a}{k} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \left(\frac{p_a}{a-1} + \frac{p_b}{b-1}\right)^{k-i} C_t(i,a) \\ &+ p_b \sum_{k=1}^{b} \binom{b}{k} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \left(\frac{p_a}{a-1} + \frac{p_b}{b-1}\right)^{k-i} C_t(i,b) \end{aligned} \tag{2.36}$$

As an example in Fig. 3 the time development of the densities on the first two layers in the case of a random tree where every vertex has two or four neighbors, is displayed.

2.4 Procedure to Derive Higher-Layer Densities

It is natural to ask whether the procedure we just described to obtain densities on the first two layers can be generalized to obtain densities in a finite number of layers. To see the issue of higher layers more clearly let us specialize from the tree to the line. In this case, we claim that the time-dependent probabilities of the occurrence of any single-site pattern describing occupations up to a given finite height can in principle be calculated. However, in most cases the (probability of occurrence of a) pattern can not be calculated directly but by a recursive algorithm which involves the computation of simpler patterns which we call the pre-image motives.

The following procedure provides a method to find the time-dependent formula of the proportion of any pattern on a vertex. It consists of four steps:

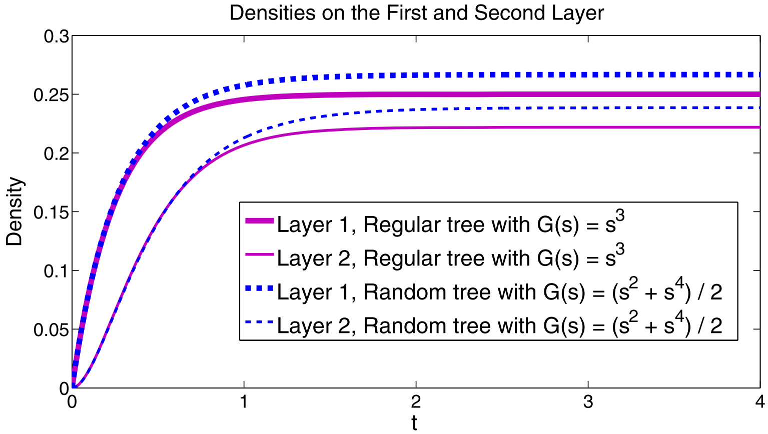

Fig. 3 Comparison of the densities on the first two layers in the case of a regular tree and a random tree

1. Find the pre-image motives (the configurations from which the pattern under interest can increase or decrease);
2. Obtain the solutions of probabilities for occurrence of the pre-image motives;
3. Construct a differential equation of the target pattern based on the pre-image motives; and finally
4. Solve the differential equation.

As an example how the program works we will now calculate the probability of the occurrence of $Y_t = (0, 1, 0, 1)'_t$, meaning the probability that the first and third layer are occupied and the second and fourth layer are empty. That the procedure stops after finitely many steps is not obvious from the beginning. Responsible for this fact is the screening. This will become clear in the example below. In a model without screening like [3] it is not true, and a corresponding recursion produces an infinite number of local motives.

2.4.1 Step 1: Find the Pre-Image Motives

In this step we have to find the patterns whose occurrences contribute to an increase or decrease of our target pattern. In the case of $Y_t = (0, 1, 0, 1)'_t$ we find four pre-image motives, i.e.

$$
\begin{aligned}
A_1 &= \begin{pmatrix} \times & \times & \times \\ 1 & 0 & \times \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \times & \times & \times \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
A_3 &= \begin{pmatrix} \times & \times & \times \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} \times & \times & \times \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\end{aligned}
\tag{2.37}
$$

This notation indicates on which position and layer a particle has been deposited (1) and where not (0), and where no particles have arrived so far ($\times$). A particle is denoted by a 1 and empty positions that will remain empty due to blocking of neighbors or to the screening effect are indicated with a 0. Positions at and beyond which no particle has arrived so far are indicated with a $\times$. Indeed, the proportion of the occurrences of Y_t will increase with the proportion of both A_1 and A_2. In both patterns a particle is able to be deposited on the third layer and complete the pattern of Y_t. The new particle can not be screened by particles in higher layers. On the other hand, the occurrence of A_3 or A_4 may lead to a decrease of Y_t, because they allow the arrival of a particle in the center location which results in $(1, 1, 0, 1)'$. There are no other motives that can directly influence the proportion of Y_t.

2.4.2 Step 2: Obtain the Solutions of the Pre-Image Motives

In this step we treat the pre-image motives one-by-one and find their solutions using the same four-step procedure again. First we look at A_1 and detect its pre-image motives.

Finding A_1

We apply the same procedure to find A_1. With an abuse of notation we write $A_1(t)$ for the probability of its occurrence.

Step 1′: The pre-image motives of A_1 are:

$$B_1 = \begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & 0 & 1 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 1 & 0 & 1 & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} \times & \times & \times \\ 1 & 0 & \times \\ 0 & 1 & 0 \end{pmatrix} \tag{2.38}$$

Step 2′: Solutions of the pre-image motives of A_1

$$B_1(t) = te^{-4t} \tag{2.39}$$

$$\begin{aligned} B_2(t) &= te^{-3t} P_t(N_t(1) = 0, m_t(1) = 1 | N_t(0) = 0) \\ &= S_t^2 te^{-3t} \\ &= te^{-4t} - te^{-5t} \end{aligned} \tag{2.40}$$

where we used our earlier result for S_t^d in (2.17).

Step 3′: The differential equation for $A_1(t)$ takes the form

$$\begin{aligned} \dot{A}_1(t) &= B_1(t) + B_2(t) - 3A_1(t) \\ &= 2te^{-4t} - te^{-5t} - 3A_1(t) \end{aligned} \tag{2.41}$$

Step 4′: Solution of $A_1(t)$

Together with $A_1(0) = 0$ we find

$$A_1(t) = \frac{7}{4}e^{-3t} - 2e^{-4t} - 2te^{-4t} + \frac{1}{4}e^{-5t} + \frac{1}{2}te^{-5t} \tag{2.42}$$

Finding A_2

We apply the same steps to get A_2.

Step 1′: The pre-image motives of A_2 are

$$C_1(t)=\begin{pmatrix}\times&\times&\times&\times\\ \times&\times&0&1\\ \times&0&1&0\end{pmatrix},\qquad C_2(t)=\begin{pmatrix}\times&\times&\times&\times\\ \times&\times&0&1\\ 1&0&1&0\end{pmatrix},\qquad A_2(t)=\begin{pmatrix}\times&\times&\times\\ 1&0&1\\ 0&1&0\end{pmatrix} \tag{2.43}$$

Step 2′: Now we solve $C_1(t)$ and $C_2(t)$. We find

$$C_1(t)=\begin{pmatrix}\times&\times&\times&\times\\ \times&\times&0&1\\ \times&0&1&0\end{pmatrix}=U_te^{-2t}.$$

With $U_t := P_t(N_t(1)=1, m_t(1)=1, N_t(2)=1, m_t(2)=2|N_t(0)=0)$. Now, we have to solve U_t first.

$$\begin{aligned}
\dot{U}_t &= -2U_t + P_t(N_t(1)=1, N_t(2)=0, N_t(3)=0|N_t(0)=0)\\
&\quad + P_t(N_t(1)=1, N_t(2)=0, N_t(3)=1, m_t(3)=1|N_t(0)=0)\\
&= -2U_t + te^{-3t}\\
&\quad + P_t(N_t(1)=1, N_t(3)=1, m_t(3)=1|N_t(0)=0, N_t(2)=0)e^{-t}\\
&= -2U_t + te^{-3t} + P_t(N_t(1)=1|N_t(0)=0, N_t(2)=0)\\
&\quad \times P_t(N_t(1)=1, m_t(1)=1|N_t(0)=0)e^{-t}\\
&= -2U_t + te^{-3t} + te^{-t}S_t^2e^{-t}\\
&= -2U_t + te^{-3t} + te^{-2t}(e^{-t}-e^{-2t})\\
&= -2U_t + 2te^{-3t} - te^{-4t}
\end{aligned} \tag{2.44}$$

So, we have to solve

$$\dot{U}_t + 2U_t = 2te^{-3t} - te^{-4t}.$$

The homogeneous solution is $U_{t,hom} = Ce^{-2t}$, and a particular solution is $U_{t,part} = -2te^{-3t} - 2e^{-3t} + \frac{1}{2}te^{-4t} + \frac{1}{4}e^{-4t}$. For $t=0$ we have $U_0 = 0$. So, this gives the general solution

$$U_t = \frac{7}{4}e^{-2t} - 2te^{-3t} - 2e^{-3t} + \frac{1}{2}te^{-4t} + \frac{1}{4}e^{-4t}.$$

Therefore, we have $C_1(t) = U_te^{-2t} = 1\frac{3}{4}e^{-4t} - 2te^{-5t} - 2e^{-5t} + \frac{1}{2}te^{-6t} + \frac{1}{4}e^{-6t}$. Now, we treat $C_2(t)$. We find

$$C_2(t)=\begin{pmatrix}\times&\times&\times&\times\\ \times&\times&0&1\\ 1&0&1&0\end{pmatrix}=U_tS_t^2e^{-t} \tag{2.45}$$

So,

$$C_2(t)=\left(1\frac{3}{4}e^{-2t} - 2te^{-3t} - 2e^{-3t} + \frac{1}{2}te^{-4t} + \frac{1}{4}e^{-4t}\right)(e^{-t}-e^{-2t})e^{-t} \tag{2.46}$$

Step 3′:

$$\dot{A}_2(t) = 2C_1(t) + 2C_2(t) - 3A_2(t)$$
$$= 7e^{-4t} - 11\frac{1}{2}e^{-5t} - 8te^{-5t} + 5e^{-6t} + 6te^{-6t} - \frac{1}{2}e^{-7t} - te^{-7t} - 3A_2(t) \quad (2.47)$$

Step 4′: So

$$A_2^h(t) = Ce^{-3t}$$

and

$$A_2^p(t) = -7e^{-4t} + \frac{31}{4}e^{-5t} + 4te^{-5t} - \frac{7}{3}e^{-6t} - 2te^{-6t} + \frac{3}{16}e^{-7t} + \frac{1}{4}te^{-7t}$$

and the general solution becomes

$$A_2(t) = \frac{67}{48}e^{-3t} - 7e^{-4t} + \frac{31}{4}e^{-5t} + 4te^{-5t} - \frac{7}{3}e^{-6t} - 2te^{-6t} + \frac{3}{16}e^{-7t} + \frac{1}{4}te^{-7t} \quad (2.48)$$

Finding A_3 and A_4

The last two motives are much less complicated compared with the former two, so we can treat them together at the same time.

Step 1′: The motives of A_3 are A_1 and A_3 itself, whereas the motives of A_4 are A_2 and A_4.
Step 2′: We already solved A_1 and A_2 above.
Step 3′: The differential equations that we need to solve are

$$\dot{A}_3(t) = A(t) - 3A_3(t) \text{ and } \dot{A}_4(t) = A_2(t) - 3A_4(t) \quad (2.49)$$

respectively.
Step 4′: With the following solutions

$$A_3(t) = -\frac{15}{4}e^{-3t} + \frac{7}{4}te^{-3t} + 4e^{-4t} + 2te^{-4t} - \frac{1}{4}e^{-5t} - \frac{1}{4}te^{-5t} \quad (2.50)$$

and

$$A_4(t) = -\frac{49}{16}e^{-3t} + \frac{67}{48}te^{-3t} + 7e^{-4t} - \frac{39}{8}e^{-5t} - 2te^{-5t}$$
$$+ e^{-6t} + \frac{2}{3}te^{-6t} - \frac{1}{16}e^{-7t} - \frac{1}{16}te^{-7t} \quad (2.51)$$

2.4.3 Step 3: Construct a Differential Equation

The differential equation for Y_t is

$$\dot{Y}_t = 2A_1(t) + A_2(t) - 2A_3(t) - A_4(t) \quad (2.52)$$

Indeed, the appearance of Y_t can increase by $A_1(t)$ and by its mirror motive. So, it counts two times. Also $A_2(t)$ increases the proportion of Y_t but only one time, because its pattern is symmetric. Decrease of Y_t occurs when a particle parks on top of $A_3(t)$ or $A_4(t)$ where the former counts two times because its mirror pattern has the same effect.

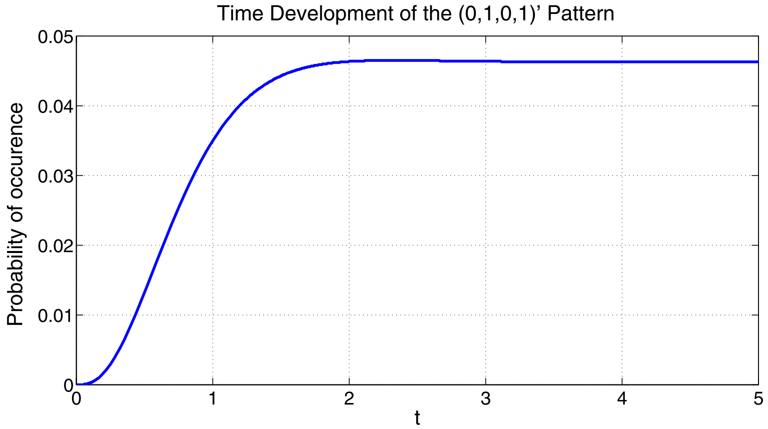

Fig. 4 Time development of the occurrence probability of $Y_t = (0, 1, 0, 1)'_t$

2.4.4 *Step 4: Solve the Differential Equation*

After some calculations we find, using $Y_0 = 0$:

$$P_t \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \frac{34}{735} - \frac{1991}{432} e^{-3t} + \frac{235}{144} t e^{-3t} + 7e^{-4t} + 2te^{-4t} - \frac{121}{40} e^{-5t} - \frac{3}{2} t e^{-5t} + \frac{17}{27} e^{-6t} + \frac{4}{9} t e^{-6t} - \frac{33}{784} e^{-7t} - \frac{5}{112} t e^{-7t} \tag{2.53}$$

Figure 4 depicts the time development of $Y(t)$.

It should be clear that the probabilities of any other pattern can be computed in a similar way.

2.5 Comparison Results

Let us now come back to the behavior on the first two layers and conclude the paper with a discussion of comparison statements for densities.

Theorem 3 *Consider a regular tree T_d. The limiting density in the first layer is higher than in the second layer for all $d \geq 2$.*

Proof Let us denote $\beta_k(d) := (d/(d-1))^d \binom{d}{k} \frac{d^{-k}}{(d-1)k+d+1}$, so $\rho^d_\infty(2) = \sum_{k=0}^d (-1)^k \beta_k(d) - \frac{d}{(d+1)^2}$. One verifies that $\beta_{k+1}(d)/\beta_k(d) < 1$ so that $k \mapsto \beta_k(d)$ is decreasing. Therefore, making use of the alternating nature of the sum, we have the bound $\lim_{t\to\infty} \rho^d_t(2) < \beta_0(d) - \beta_1(d) + \beta_2(d) - d/(d+1)^2$. The proof is then concluded by seeing that $\beta_0(d) - \beta_1(d) + \beta_2(d) < (2d+1)/(d+1)^2$ for all $d \geq 2$. After some algebraic manipulations we find $\beta_0(d) - \beta_1(d) + \beta_2(d) = (d/(d-1))^d \frac{2(d-1)}{(d+1)(3d-1)}$. So, we have to check that $(d/(d-1))^d \frac{2(d-1)}{3d-1} < \frac{2d+1}{d+1}$ or equivalently $(1 - \frac{1}{d})^d > \frac{2(d-1)(d+1)}{(3d-1)(2d+1)}$. Developing the left term into a series and truncating it, we also find that $(1 - \frac{1}{d})^d \geq \frac{d-1}{2d} - \frac{(d-1)(d-2)}{6d^2} = \frac{2(d-1)(d+1)}{6d^2}$. Equality holds only in the cases of $d = 2$ and $d = 3$. Furthermore, it is clear that $\frac{2(d-1)(d+1)}{6d^2} > \frac{2(d-1)(d+1)}{(3d-1)(2d+1)}$

for $d \geq 2$. So, finally, by checking the cases $d = 2$ and $d = 3$ directly we conclude that the density of the second layer is strictly dominated by the first layer density for all $d \geq 2$. $\square$

In [6] the issue of comparing the behavior of the process on a regular tree with that on a random tree having the same number of nearest neighbors on the average was raised, and a number of results were given. In our situation, we have the following.

Theorem 4 *Consider the random trees S and T with probability generating functions G_S and G_T respectively. If $G_S(s) > G_T(s)$, for all $0 < s < 1$ then the first layer density of S exceeds the first layer density of T for all $t > 0$.*

In particular, the first layer density of the regular tree $T_{d'}$ dominates the first layer density of the regular tree T_d for all $t > 0$ if $d' < d$.

Proof According to Theorem 2 the density of the first layer on a random tree S with probability generating function $G(s) = \sum_{n=2}^{\infty} a_n s^n$ is given by $\mathbb{Q}\rho_t^S(1) = \sum_{k=2}^{\infty} a_k \frac{1-e^{-(k+1)t}}{k+1}$. Define $\gamma(t) := \mathbb{Q}\rho_t^S(1) - \mathbb{Q}\rho_t^T(1) = \sum_{k=2}^{\infty}(a_k - b_k)(\frac{1-e^{-(k+1)t}}{k+1})$.We have $\gamma(0) = 0$. In case of $G_S(s) > G_T(s)$, the time derivative of $\gamma(t)$ becomes $\frac{d}{dt}\gamma(t) = \sum_{k=2}^{\infty}(a_k - b_k)e^{-(k+1)t} = e^{-t}(G_S(e^{-t}) - G_T(e^{-t})) > 0$ for all $t > 0$. $\square$

Corollary 1 *Consider a regular tree T_d with d neighbors for each vertex, with $d \in \{2, 3, 4, \dots\}$. If random tree S has an average number of d vertex neighbors, then for any $t > 0$, the density of the first layer on S is higher than on T_d.*

Proof Consider T_d with generating function $G_T(s) = s^d$ and random tree S with average vertex neighbors $EX = d$. From Jensen's inequality it follows that $G_S(s) = E(s^X) > s^{EX} = s^d = G_T(s)$, because $f(z) = a^z$, $(a > 0)$ is a convex function. Application of Theorem 4 completes the proof. $\square$

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